ELSEVIER Journal of Geometry and Physics 20 (1996) 19–30

JOURNAL OF GEOMETRY AND PHYSICS

Composite link polynomials from super Chern–Simons theory

Olaf J. Backofen [1]

Institut für Theoretische Physik, Universität Saarbrücken, FR. 10.1, D-66041 Saarbrücken, Germany

Received 31 May 1995

Abstract

Using the framework of supersymmetric Witten–Jones theory the composite link polynomials related to the basic classical simple complex Lie superalgebras will be computed. The related graded Casimir operators will be given explicitly for arbitrary covariant class I representations. As a consequence of the topological interpretation of link invariants, it is essentially possible to derive the Boltzmann weights of the associated IRF models found previously as solutions of the graded Yang–Baxter equation.

Subj. Class.: Quantum field theory
1991 MSC: 57M25, 17B70, 57N10, 82B23
Keywords: Composite knot invariants; Chern–Simons theory; Lie superalgebras

1. Introduction

The universality of Witten–Jones theory has been discussed in many papers [1–15]. Using the framework of topological Chern–Simons theory (CST) a great variety of previously known but also new link invariants was derived. Previously, a host of intimate connections with statistical mechanics and rational conformal field theories was established [16–21]. For example, Akutsu–Wadati knot invariants derived earlier from solvable models related to the Yang–Baxter equation (YBE) [22–25] could be given as vacuum expectation values of gauge invariant Wilson loop operators associated with certain representations of simple Lie groups [3–8].

[1] E-mail: ph11ahob@sbusol.rz.uni-sb.de.

SSDI 0393-0440(95)00044-5

It was Horne [26] to extend Witten's pioneering work [1] to supersymmetric Lie groups [27–29], calculating link polynomials related to elementary representations of the basic graded Lie groups. However, it is possible to extend Horne's approach to arbitrary covariant class I representations in order to introduce composite supersymmetric link invariants. In analogy with the non-graded case it is then feasible to rederive powerful knot polynomials related to solutions of the graded Yang–Baxter equation (GYBE) [30,31] by topological means within the CST and point out further generalizations. In this context the topological derivation is of particular interest since the well-known fusion method of Wadati et al. is applicable only for solvable models with quadratic minimal polynomials such as interaction-round-a-face (IRF) models associated with $SU(r|s)$ [23,30].

Calculating the related graded Casimir operators explicitly, it is not only practicable to evaluate the knot invariants but also the relevant Boltzmann weights of the associated solvable models found previously as solutions of the GYBE [30].

The paper is organized as follows: Section 2 will review Witten–Jones theory for supersymmetric Lie groups and give explicitly the corresponding Casimir operators for arbitrary covariant class I representations. The supersymmetric composite invariants will be examined in Section 3 in a case-by-case study. A detailed conclusion and an outlook to further generalizations will be provided in Section 4.

2. Super Witten–Jones theory

The basic objects of Witten–Jones theory are the vacuum expectation values on the three-sphere S^3 of Wilson line operators in Chern–Simons gauge theory based on an arbitrary simple Lie group G [1]. A link L in S^3 may then be considered as a disjoint union of circles C_i, oriented and labeled by a choice of representations R_i of G. The relevant link expectation value is given by

$$\langle L \rangle = Z_{G,R}(M, L) = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}\mathcal{L}_{\mathrm{CS}}} \prod_i W_{R_i}(C_i), \tag{1}$$

where $\mathcal{L}_{\mathrm{CS}}$ is the Chern–Simons Lagrange action and $\{W_{R_i}(C_i)\}$ are the gauge invariant Wilson line operators [1].

Following Horne [26] it is possible to extend Witten–Jones theory to include complex simple Lie superalgebras of Kac type $A(r|s)$, $B(r|s)$, $C(s)$ and $D(r|s)$ [27–29], using the supersymmetric CS action

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi} \int_M str(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A). \tag{2}$$

The gauge connection A is now given by a supermatrix containing bosonic and fermionic parts and the supertrace str guarantees the supergroup invariance. Naturally, we will have to exclude Lie superalgebras with degenerate Killing form such as $A(r|r)$, $D(s+1|s)$ and $D(2|1, \alpha)$, since it does not seem to be possible to define the CS action in these cases. A detailed study of super Chern–Simons theory was presented by Sakai and Tanii [32].

In analogy with the non-graded case, $Z_{G,R}(M, L)$ can be computed employing an algorithm for untangling knots [1]. Quantization of the super Chern–Simons theory in the presence of Wilson lines again defines a physical Hilbert space $\mathcal{H}$ associated with the choice of representations R_i of G. Likewise the dimension N of $\mathcal{H}$ is given once more from the decomposition of the chosen representation R of the super Lie group G into irreducible representations E_i:

$$R \otimes R = \bigoplus_{i=1}^{N} E_i. \tag{3}$$

However, in the case of graded Lie groups finite-dimensional representations are not necessarily completely reducible, defining the so-called atypical representations, and we have to distinguish tensor products of both covariant and contravariant basis vectors of the same class, as well as tensor products of class I and class II representations [33–35]. Nevertheless, in the context of deriving graded knot invariants it is essentially possible to restrict on pure covariant class I representations. In this case the super Young tableaux correspond to ordinary Young tableaux [34] and the associated quadratic Casimir operators seperate the irreducible representations completely [36].

The highest weight Λ of the irreducible representation under consideration is usually expressed as a sum of rank G fundamental weights Λ_j,

$$\Lambda = \sum_{j=1}^{\operatorname{rank} G} a_j \Lambda_j, \tag{4}$$

defining the Kac–Dynkin coefficients $\{a_j\}$ of the concerned representation Γ. The fundamental weights are calculated as usually from the inverse of the Cartan matrices and the simple roots of G, which are given for example in [37]. The quadratic Casimir operator C_Γ may now be evaluated similarly as in the non-graded case using

$$C_\Gamma = \langle \Lambda | \Lambda + 2\rho \rangle, \tag{5}$$

where now the dual Weyl vector has to be determined from the bosonic and fermionic roots (cf. [28])

$$\rho = \frac{1}{2} \left[\sum_{\alpha_i \in \Delta_0^+} \alpha_i - \sum_{\alpha_i \in \Delta_1^+} \alpha_i \right]. \tag{6}$$

Considering the standard normalization of the fundamental weights and the distinguished choice of simple roots, the graded quadratic Casimir operators for the classical simple complex Lie algebras are found to be explicitly for:

$$
\begin{aligned}
A(r|s):\ C_\Gamma = \frac{1}{2}\Bigg[&\sum_{i=1}^{r+s+1}\left(\sum_{j=i}^{r+s+1} a_j\left(\sum_{j=i}^{r+s+1} a_j - 2i\right)\right) + (r-s+1)\sum_{j=i}^{r+s+1} ja_j \\
&- \frac{1}{r-s}\left(\sum_{j=i}^{r+s+1} ja_j\right)^2\Bigg] \quad (r > s \geq 0),
\end{aligned}
$$

$$
\begin{aligned}
B(r|s):\ C_\Gamma = \frac{1}{2}\Bigg[&\sum_{i=1}^{r+s}\left(\sum_{j=i}^{r+s} a_j - \frac{a_{r+s}}{2}\right)\left(\sum_{j=i}^{r+s} a_j - \frac{a_{r+s}}{2} + 2r - 2s - 2i + 1\right)\Bigg] \\
&(r \geq 0, s \geq 1),
\end{aligned}
$$

$$
C(s):\quad C_\Gamma = \frac{1}{2}\left[\sum_{i=1}^{s}\left(\sum_{j=i}^{s} a_j - \frac{a_s}{2}\right)\left(\sum_{j=i}^{s} a_j - \frac{a_s}{2} - 2s - 2i + 4\right)\right] \quad (s \geq 2),
$$

$$
\begin{aligned}
D(r|s):\ C_\Gamma = \frac{1}{2}\Bigg[&\sum_{i=1}^{r+s}\left(\sum_{j=i}^{r+s} a_j - \frac{a_{r+s}}{2}\right)\left(\sum_{j=i}^{r+s} a_j - \frac{a_{r+s}}{2} + 2r - 2s - 2i\right)\Bigg] \\
&(r \geq 2, s \geq 1, r \neq s+1).
\end{aligned}
\tag{7}
$$

The Casimir elements may be determined alternatively, in particular in the case of $A(r|s)$, from the results of Baha Balantekin [38], Jarvis and Green [39] by taking into account that $\sum_{j=1}^{\operatorname{rank} G} ja_j$ represents the total number of boxes in the Young tableau and $\sum_{j=i}^{\operatorname{rank} G} a_j = n_i$ corresponds to the Kac–Dynkin labeling $\{n_i\}$ of the representation.

We may now proceed as in the case of non-graded Witten–Jones theory. Using the well-known connection between the physical Hilbert spaces in (2+1)-dimensions and the spaces of conformal blocks in (1+1)-dimensions, it is feasible to determine the skein relation

$$
\begin{aligned}
&\alpha_- Z_{G,R}(M, L_-) + \alpha_0 Z_{G,R}(M, L_0) + \alpha_+ Z_{G,R}(M, L_+) \\
&\quad + \cdots + \alpha_{(N-1)+} Z_{G,R}(M, L_{(N-1)+}) = 0
\end{aligned}
\tag{8}
$$

from the eigenvalues λ_i of the corresponding monodromy matrices:

$$
\begin{aligned}
\alpha_- &= (-1)^N \prod_{i=1}^{N} \lambda_i, \\
\alpha_0 &= (-1)^{N-1}\left(\prod_{i=1}^{N} \lambda_i\right)\left(\sum_{i=1}^{N} \lambda_i^{-1}\right), \\
&\ \vdots \\
\alpha_{(N-2)+} &= -\sum_{i=1}^{N} \lambda_i, \\
\alpha_{(N-1)+} &= \ 1.
\end{aligned}
\tag{9}
$$

As usual, the monodromy eigenvalues λ_i can be deduced from the conformal weights of the primary conformal field transforming as the representations involved, R and E_i:

$$\lambda_i = (-1)^{N+i} \exp[\mathrm{i}\pi(2h_R - h_{E_i})]. \tag{10}$$

Here the conformal weights h_Γ may be calculated from the Casimir operator of the adjoint representation C_v and the quadratic Casimir elements C_Γ given in (7):

$$h_\Gamma = \frac{1}{k + C_v} C_\Gamma. \tag{11}$$

Selecting now the spin-s representation of the classical simple complex Lie superalgebras the corresponding composite link polynomials are obtained straightforwardly as shown in Section 3.

It is important to observe that we may express the Boltzmann weights of the associated exactly solvable models in terms of the braiding matrix B_k at the face k on a primary field a and the corresponding monodromy eigenvalues λ_i given in (10). According to Gepner [20] the projection operator associated to B_k is defined by

$$P^a = \prod_{i=1, i\neq a}^{N} \frac{B_k - \lambda_i}{\lambda_a - \lambda_i}. \tag{12}$$

The corresponding IRF model may then be introduced via its Boltzmann weights described in the usual operator form (cf. [20])

$$X_k(u) = \sum_{a=1}^{N} P_k^a f^a(u) \tag{13}$$

with the functions

$$f^a(u) = \prod_{i=1}^{a-1} \sin(\zeta_i + u) \prod_{i=1}^{N-1} \sin(\zeta_i - u), \tag{14}$$

where

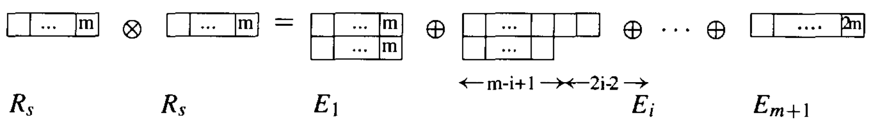
$$\zeta_i = \tfrac{1}{2}\pi[h_{E_{i+1}} - h_{E_i}]. \tag{15}$$

Here u is the spectral parameter which labels the family of models. The braiding matrix generally is computed from the quantum Clebsch–Gordon coefficients [40] and restricting, for example, to the fundamental and spin-s representation of $A(r|s)$, the results agree with the Boltzmann weights of the $SU(r+1|s+1)$-IRF model and IRF fusion model, respectively, introduced by Deguchi et al. [31] in terms of Jacobi elliptic functions. Naturally, the Boltzmann weights of OSp IRF fusion models can be constructed explicitly in the same way. Hereby the resulting IRF models are generalizations of integrable spin chains consisting of bosons and fermions introduced originally by Sutherland [41].

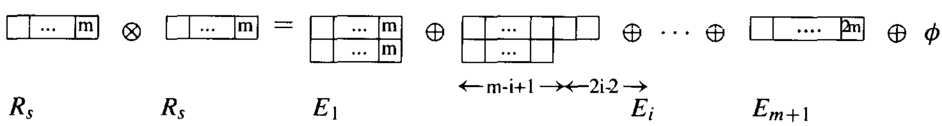
3. The supersymmetric composite invariants

In order to examine the properties of the supersymmetric composite invariants in a systematic way it is appropriate to evaluate the relevant link polynomials explicitly. Like in the non-graded case it follows from the properties of RCFT and the corresponding IRF models defined by (13), that the invariants so constructed always obey the Markov properties and thus are well defined [6,20].

Composite link invariants are given by Witten's path integral associated with the irreducible representations in the product of the spin s representation R_s placed on the individual $m = 2s$ strands of a composite braid [6,8,10]. When restricting to covariant class I representations, R_s is given by Young tableaux containing m boxes in a row, in analogy with the non-graded case. The decomposition relation (3) reads as:



for $A(r|s)$,



for $B(r|s)$, $C(s)$ and $D(r|s)$. (16)

The Kac–Dynkin coefficients $\{a_i\}$ of the irreducible representations R_s, E_i, and ϕ are $\{m, 0, \ldots, 0\}$, $\{2i-2, m-i+1, 0, \ldots, 0\}$, and $\{0, \ldots, 0\}$, respectively. Referring to the results of the previous section the associated Hilbert spaces are of dimension $N = (m+1)$ and $N = (m+2)$ (cf. (3)), leading to an $(m+1)$ and $(m+2)$-dimensional skein relation (8) for the cases of $A(r|s)$ and $B(r|s)$, $C(s)$, $D(r|s)$, respectively. Observing that a realization of $A(r|s)$ is provided by $SU(r+1|s+1)$, the Casimir elements corresponding to (16) may now be calculated from (7) for the various classical simple Lie superalgebras ($i = 1, \ldots, m+1$):

$$A(r-1|s-1):\ C_{R_s} = \frac{1}{2}\left[m^2 - m + m(r-s) - \frac{m^2}{r-s}\right],$$

$$C_{E_i} = m^2 - 2m + i^2 - i + m(r-s) - \frac{2m^2}{r-s},$$

$$B(r|s):\quad C_{R_s} = \tfrac{1}{2}[m^2 - m + 2m(r-s)],$$
$$C_{E_i} = m^2 - 2m + i^2 - i + 2m(r-s),$$

$$C(s):\quad C_{R_s} = \tfrac{1}{2}[m^2 + 2m + 2ms],$$
$$C_{E_i} = m^2 - 2m + i^2 - i + m(3-2s),$$

$$D(r|s):\quad C_{R_s} = \tfrac{1}{2}[m^2 - 2m + 2m(r-s)],$$
$$C_{E_i} = m^2 - 2m + i^2 - i + 2m\left(r - s - \tfrac{1}{2}\right). \tag{17}$$

The skein relation coefficients (9) for the composite link invariants associated with the spin-$s = \frac{1}{2}m$ representation may now be determined straightforwardly from the eigenvalues λ_i (10) for the different cases of supersymmetric Lie groups. The usual q variable substitution

$$q = \exp\left[\frac{2\pi \mathrm{i}}{k + C_v}\right] \tag{18}$$

imposes simultaneously the deformation parameter of quantum groups related to primary fields of Wess–Zumino conformal theory [42,43]. In order to reach agreement with the notation used in knot literature based on standard framing, as usually, the Wilson lines must be adjusted by j-fold Dehn twists [1], imposing a subsequent multiplication of the coefficients α_{j+} by $\exp[-j\pi \mathrm{i} h_{R_s}]$ ($j = 0, \ldots, N-1$).

The case of ${}^m A(r-1|s-1)$ ($SU(r|s)$): The eigenvalues $\{\lambda_i\}$ are given by ($i = 1, \ldots, m+1$)

$$\lambda_i = (-1)^{m+1+i}\, q^{[-(1/2)\,i(i-1) + (m(m+(r-s))/2(r-s))]} \tag{19}$$

leading to an $(m+1)$-dimensional skein relation as in the case of the non-graded Lie groups ${}^m A_{n-1}$. For instance, in the non-composite case, i.e., ${}^1 A(r-1|s-1)$, we will obtain the supersymmetric version of the HOMFLY polynomial (cf. [26])

$$q^{-(r-s)/2} L_+ + (q^{1/2} - q^{-1/2}) L_0 - q^{(r-s)/2} L_- = 0; \tag{20}$$

the case ${}^m A(r-1|s-1)$ yields the supersymmetric extension of the generalized (2-variable) Akutsu–Wadati polynomial (cf. [6]), i.e., for $m = 2$

$$L_{2+} + (-q^{r-s+2} + q^{r-s+1} + q^{r-s-1}) L_+$$
$$+ (-q^{2(r-s)+3} - q^{2(r-s)} + q^{2(r-s)+1}) L_0 + q^{3(r-s)+2} L_- = 0. \tag{21}$$

These results agree exactly with the original defining relations of the composite invariants of Deguchi et al. [30], derived from solutions of the GYBE using the fusion method. Comparing (19)–(21) with the corresponding results of the non-graded theory we observe that it is possible to obtain the invariants related to $A(r-1|s-1)$ from those of A_{n-1} by the replacement $n \to r - s$ (as stated already by Horne [26] and Riggs et al. [44] for the non-composite case). Thus for $m = 1$ the theory yields only one in this sense new invariant corresponding to $r - s = 1$ (recalling that $r - 1 > s - 1 \geq 0$), whereas the composite case implies m different link polynomials. As shown by Kauffman and Saleur [45] the Alexander–Conway polynomial is related to the ${}^1A(r|r)$ algebras excluded in the Chern–Simons approach. However, if we relax the defining conditions of $A(r|s)$ in order to obtain the graded Lie algebra $gl(r|s)$ it seems to be possible to include the case $r = s$.

The case of ${}^mB(r|s)$ $(OSp(2r+1|2s))$: The eigenvalues of the ${}^mB(r|s)$ monodromy matrices are found to be $(i = 1, \ldots, m+1)$:

$$\lambda_i = (-1)^{i+m+1} q^{-i^2/2+i/2+m/2} \quad \text{and} \quad \lambda_\phi = q^{(m/2)[m+2(r-s)-1]}. \tag{22}$$

Recalling that $h_\phi = 0$, the eigenvalues related to the scalar representation follow from the conformal weights of the spin-s representation R_s (cf. (10)). In analogy with the non-graded case the skein relation is of degree $(m+2)$ and the link polynomials related to ${}^mB(r|s)$ again may be obtained from the corresponding results of ${}^mB_n$ by the replacement $n \to r - s$. However, now applying $r \geq 0$ and $s \geq 1$, yields a hierarchy of m different polynomials for each value of $r - s = 0, -1, -2 \ldots$, corresponding for $s > r$ to a Witten–Jones theory with gauge group ${}^mB(s|r)$ at level $-k$ (cf. [26]). Kauffman polynomials correspond to ${}^1B_n$ non-graded Chern–Simons theory. Observe now that the link polynomials obtained this way agree with the results of Zhang et al. [46–48], where the theory of quantum supergroups is applied to the construction of braid group representations and link invariants. For example, in the case of ${}^mB(0|1)$, provided by $OSp(1|2)$, the skein relation is given by

$$\prod_{i=1}^{m+1} q^{-i^2/2+i/2+m/2} L_- + \cdots + q^{-(1/4)[m^3-2m^2-3m]} L_{(m+1)+} = 0. \tag{23}$$

This coincides (modulo the framing convention) with the skein relation of the link polynomials derived from the spin-s representation over the quantum group $U_q(OSp(1|2))$ by Zhang [48] (q being a root of unity) given there in the form

$$L(\hat{\theta}) = q^{-m((m/2)+2)\sum_{s=1}^{\mathcal{N}-1} l_s} \times \prod_{t=1}^{\mathcal{N}-1} \sum_{i=0}^{m} (-1)^{il_t} q^{(m-i+2)(m-i)(l_t/2)} \frac{q^{m-i+1/2} + q^{-m+i-1/2}}{q^{m/2+1/2} + q^{-m/2-1/2}}$$

for the closed braid $\hat{\theta} = (b_{i_1})^{l_1}(b_{i_2})^{l_2} \cdots (b_{i_{\mathcal{N}-1}})^{l_{\mathcal{N}-1}}$ of the braid group $\mathcal{B}_\mathcal{N}$. Hereby the skein relation (23) may be obtained as usually from the $(m+2)$th order polynomial equation

$$\sum_{j=1}^{m+2} (G_i^j - \lambda_j) = 0 \tag{24}$$

for the braid element G_i^j, described, e.g., by Gepner [20] or Akutsu and Wadati [49].

The case of ${}^mC(s)$ $(OSp(2|2s-1))$: The $(m+2)$-dimensional skein relation may be determined from the eigenvalues $\{\lambda_i\}$ $(i = 1, \ldots, m+1)$:

$$\lambda_i = (-1)^{i+m+1} q^{-i^2/2+i/2+m/2} \quad \text{and} \quad \lambda_\phi = -q^{(m/2)[m+2(1-s)]}. \tag{25}$$

Thus the Chern–Simons approach yields once more similiar link polynomials as in the composite non-graded case, featuring however different skein relation coefficients for all possible values of s, i.e., $s \geq 2$.

The case of ${}^mD(r|s)$ $(OSp(2r|2s))$: Naturally, also the ${}^mD(r|s)$ invariants obey an $(m+2)$-dimensional skein relation as in the case of ordinary Lie groups. The eigenvalues are given by $(i = 1, \ldots, m+1)$:

$$\lambda_i = (-1)^{i+m+1} q^{-i^2/2+i/2+m/2} \quad \text{and} \quad \lambda_\phi = q^{(m/2)[m+2(r-s)-2]}, \tag{26}$$

coinciding with the corresponding results of ${}^mD_n$, after the replacement $n \to r - s$. Recalling that $r \geq 2$, $s \geq 1$ and $r \neq s + 1$, we will obtain new skein relation coefficients for the condition $s \geq r$.

4. Conclusion and outlook

The main objective of the present work is given by the incorporation of supersymmetry in Witten's original quantum field theoretical approach to knot theory in order to derive composite link polynomials defined earlier from solutions of the GYBE, using the fusion method. Hereby the primary intention is to present explicitly neglected results in the latter area and to outline in a pedagogical way the computation of a variety of known and new link polynomials. For instance, the HOMFLY polynomial and Akutsu–Wadati polynomials are derived as special cases, when restricting to the non-graded theory. The field theoretical interpretation is of particular importance for the super Lie groups $B(r|s)$, $C(s)$ and $D(r|s)$ since the fusion method fails to derive the corresponding composite link invariants.

We may observe the close relationship between the results of graded and non-graded theories which is a natural consequence of the definition of supersymmetric Lie groups. While in the case of non-composite theory we obtain new skein relation coefficients only for a few values of the group dimensions $\{r, s\}$, i.e., for example in the case of $A(r|s)$ for the values $r - s = 1$, the composite case implies a hierarchy of m different link polynomials.

Naturally, also the supersymmetric Witten–Jones theory related to composite representations is not unitary in general, which is reflected in the appearance of negative values of the expectation value of the unknot $\langle \bigcirc \rangle$, e.g., in the case of ${}^mC(s)$ (cf. Horne [26]).

It is interesting to observe that the rank–level duality of CST is reflected in the supersymmetric link invariants. Since the permutation of the monodromy eigenvalues λ_i does not lead to new results of the skein relation coefficients, the same link invariants will be obtained for dual decompositions of the chosen representation (see [6,44]).

Higher-spin polynomials being progressively more powerful, emphasize the importance of the introduction of composite link invariants. For instance, as shown by Ramadevi et al. [50], the chirality of the two knots 9_{42} and 10_{71} is not detected by any of the well-known polynomials, namely Jones, HOMFLY and Kauffman. However, the composite ${}^mA_{n-1}$ invariants are indeed sensitive to the chirality of these knots for $m \geq 3$. In this sense the supersymmetric approach represents a further step of generalization.

Unfortunately, any of the derived composite knot invariants do not distinguish isotopically inequivalent mutant knots and links. Recently, Ramadevi et al. [51] succeeded in the construction of r-composite invariants given by a sum of ordinary Chern–Simons invariants placing a set of r-representations in each composite strand. As a consequence some mutant links can be distinguished and it might be interesting to examine the corresponding extension to supersymmetric Lie groups. Furthermore, we may also study multicolored graded invariants, where different representations are placed on different composite strands.

The present approach was restricted to cases of the classical simple complex Lie superalgebras. A generalization to the exceptional Lie superalgebras $F(4)$ and $G(3)$ is straightforward, whereas it seems to be impossible to construct a Witten–Jones theory for the strange superalgebras $P(n)$ and $Q(n)$. However, Frappat et al. [52] succeeded in the construction of a non-degenerate Killing form, simple root systems and highest weight irreducible representations for $P(n)$, implying eventually a possibility for the derivation of $P(n)$ invariants.

Further generalizations may be discovered when exploring the derivation of knot invariants related to the $Z_2 \oplus Z_2$ graded color superalgebras (see Jarvis et al. [53]) as, e.g., $SpO(2r|1|2s|0)$.

Finally, we may examine the construction of quantum universal enveloping algebras using super Chern–Simons theory. In the same way Witten succeeded in deriving the corresponding structure coefficients starting only from the general covariance of 3D Chern–Simons theory with gauge group $SU(2)$ [43], it is essentially possible to generalize Witten's approach to arbitrary classical Lie groups as, e.g., in the case of link invariants.

It is important to emphasize that the main results within the present approach proceed directly from Witten's Chern–Simons theory and thus are not mathematically rigorous [1], however, they can be verified by other means such as the quantum group approach. For example, in the case of ${}^{m}B(0|1)$ the results agree with the link polynomials related to the quantum group $U_q(OSp(1|2))$ derived by Zhang et al. [48]. As a consequence it is instructive to compute composite link polynomials also from super Chern–Simons theory in order to compensate the lack of analogy with Witten–Jones theory.

Acknowledgements

It is a pleasure to thank A. Holz, L.H. Kauffman, V.G. Turaev and S. Schuler for several useful discussions. Furthermore I have benefitted from helpful advice via e-mail from J. Fuchs, M. Scheunert and E. Witten. This work was partially supported by the Landesgraduiertenförderung of the Saarland.

References

[1] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1989) 351.
[2] Y. Wu and K. Yamagishi, Chern–Simons theory and Kauffman polynomials, Internat. J. Modern Phys. A 5 (1990) 1165.
[3] M.-L. Ge, Y.-Q. Li and K. Xue, Extended state expansions and the universality of Witten's version of link polynomial theory, Internat. J. Modern Phys. A 5 (1990) 1975.

[4] M.-L. Ge, Y.-Q. Li and K. Xue, Extended state model and group approach to new polynomials, J. Phys. A 23 (1990) 619.
[5] O.J. Backofen, Quantum field theory and A,B,C,D IRF model invariants, in: *Low-Dimensional Topology and Quantum Field Theory*, ed. H. Osborn (Plenum, New York, 1993) pp. 19–29.
[6] O.J. Backofen, Composite A,B,C,D IRF model invariants, J. Geom. Phys. 19 (1996) 123.
[7] J.M. Isidro, J.M.F. Labastida and A.V. Ramallo, Polynomials for torus links from Chern–Simons gauge theories, Nucl. Phys. B 398 (1993) 187.
[8] R.K. Kaul and T.R. Govindarajan, Three-dimensional Chern–Simons theory as a theory of knots and links, Nucl. Phys. B 380 (1992) 293.
[9] R.K. Kaul and T.R. Govindarajan, Three-dimensional Chern–Simons theory as a theory of knots and links (II. Multicoloured links), Nucl. Phys. B 393 (1993) 392.
[10] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, Three-dimensional Chern–Simons theory as a theory of knots and links (III. Compact semi-simple group), Nucl. Phys. B 402 (1993) 548.
[11] R.K. Kaul, Chern–Simons theory, coloured-oriented braids and link invariants, IMSc/93/3 preprint, Madras University (1993).
[12] M. Hayashi, Knot and link polynomials for compact simple Lie algebras, YITP/U-91-39 preprint, Kyoto University (1991).
[13] P. Cotta-Ramusino, E. Guadagnini, M. Martellini and M. Mintchev, Quantum field theory and link invariants, Nucl. Phys. B 330 (1990) 557.
[14] E. Guadagnini, The universal link polynomial, Internat. J. Modern Phys. A 7 (1992) 877.
[15] E. Guadagnini and L. Pilo, Reduced tensor algebra in $SU(3)$ Chern–Simons field theory, J. Geom. Phys. 14(1994)236; $SU(3)$ Chern–Simons field theory in three manifolds, J. Geom. Phys. 14(1994)365.
[16] L.H. Kauffman, *Knots and Physics* (World Scientific, Singapore, 1991).
[17] L.H. Kauffman, State models and the Jones polynomial, Topology 26 (1987) 395.
[18] L.H. Kauffman, Statistical mechanics and the Jones polynomial, AMS Contemp. Math. Ser. 78 (1989) 263.
[19] C.N. Yang and M.-L. Ge, *Braid Group, Knot Theory and Statistical Mechanics* (World Scientific, Singapore, 1989).
[20] D. Gepner, Foundations of rational conformal field theory (I.), CALT 68-1825 preprint; On solvable lattice models and knot invariants CALT 68-1870 preprint, California Institute of Technology (1993).
[21] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, Knot invariants from rational conformal field theories, IMSc/93/50 preprint, Madras University (1993).
[22] V.F. Jones, On knot invariants related to some statistical mechanical models, Pacific J. Math. 137 (1989) 311.
[23] M. Wadati, T. Deguchi and Y. Akutsu, Exactly solvable models and knot theory, Phys. Rep. 180 (1989) 247.
[24] M. Wadati, Y. Yamada and T. Deguchi, Knot theory and conformal field theory: reduction relations for braid generators, J. Phys. Soc. Japan 58 (1989) 1153.
[25] V.G. Turaev, The Yang–Baxter equation and invariants of links, Invent. Math. 92 (1988)527.
[26] J.H. Horne, Skein relations and Wilson loops in Chern–Simons gauge theory, Nucl. Phys. B 334 (1990)669.
[27] V.G. Kac, Lie superalgebras, Adv. in Math. 26 (1977) 8.
[28] J.F. Cornwell, *Group Theory in Physics III* (Academic Press, New York, 1989).
[29] P. Ramond, Supersymmetry in physics: an algebraic overview, Physica D 15 (1985) 25.
[30] T. Deguchi and Y. Akutsu, Graded solutions of the Yang–Baxter relation and link polynomials, J. Phys. A 23(1990)1861.
[31] T. Deguchi and A. Fujii, IRF models associated with representations of the Lie superalgebras $gl(m|n)$ and $sl(m|n)$, UT-Komaba 91-4 preprint, Tokyo University (1991).
[32] N. Sakai and Y. Tanii, Super Wess–Zumino–Witten models from super Chern–Simons theories, Progr. Theoret. Phys. 83 (1990) 968.
[33] V.G. Kac, Representations of classical Lie superalgebras, in: *Differential Geometric Methods in Mathematical Physics*, eds. K. Bleuler, H.R. Petry and A. Reeds (Springer, Berlin, 1978) pp. 597–626.
[34] A. Baha Balantekin and I. Bars, Dimension and character formulas for Lie supergroups, J. Math. Phys. 22 (1981) 1149; Representations of supergroups, J. Math. Phys. 22 (1981) 1810.

[35] P.H. Dondi and P.D. Jarvis, Diagram and superfield techniques in the classical superalgebras, J. Phys. A 14 (1981) 547.
[36] M. Scheunert, Casimir elements of Lie superalgebras, in: *Differential Geometric Methods in Mathematical Physics*, ed. S. Sternberg (Reidel, Dordrecht, 1984) pp. 115–124.
[37] F.W. Lemire and J. Patera, Congruence classes of finite representations of simple Lie superalgebra, J. Math. Phys. 23 (1982) 1409.
[38] A. Baha Balantekin, Anomalies and eigenvalues of Casimir operators for Lie groups and supergroups, J. Math. Phys. 23 (1982) 486.
[39] P.D. Jarvis and H.S. Green, Casimir invariants and characteristic identities of the general linear, special linear and orthosymplectic graded Lie algebras, J. Math. Phys. 20 (1979) 2115.
[40] Z.-Q. Ma, *Yang–Baxter Equation and Quantum Enveloping Algebras* (World Scientific, Singapore, 1993).
[41] B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B 12 (1975) 3795.
[42] E. Witten, Gauge theories and integrable lattice models, Nucl. Phys. B 322 (1989) 629.
[43] E. Witten, Gauge theories, vertex models, and quantum groups, Nucl. Phys. B 330 (1990) 285.
[44] M. Bourdeau, E.J. Mlawer, H. Riggs and H.J. Schnitzer, The quasi-rational fusion structure of $SU(m|n)$ Chern–Simons and WZW theories, Nucl. Phys. B 372 (1992) 303.
[45] L.H. Kauffman and H. Saleur, Free fermions and the Alexander–Conway polynomial, Comm. Math. Phys. 141 (1991) 293.
[46] R.B. Zhang, Graded representations of the Temperley–Lieb algebra, quantum supergroups, and the Jones polynomial, J. Math. Phys. 32 (1991) 2605.
[47] R.B. Zhang, M.D. Gould and A.J. Bracken, Quantum group invariants and link polynomials, Comm. Math. Phys. 137 (1991) 13.
[48] R.B. Zhang, Braid group representations arising from quantum supergroups with arbitrary q and link polynomials, MRR-006-92 preprint, Australian National University Canberra (1992).
[49] Y. Akutsu and M. Wadati, Exactly solvable models and new link polynomials (I. N-State vertex models), J. Phys. Soc. Japan 56 (1987) 3039.
[50] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, Chirality of knots 9_{42} and 10_{71} and Chern–Simons theory, IMSc/93/56 preprint, Madras University (1993).
[51] P. Ramadevi, T.R. Govindarajan and R.K. Kaul, Representations of composite braids and invariants for mutant knots and links in Chern–Simons field theories, IMSc/94/45 preprint, Madras University (1994).
[52] L. Frappat, A. Sciarrino and P. Sorba, Dynkin-like diagrams and representations of the strange superalgebra $P(n)$, J. Math. Phys. 32 (1991) 3268.
[53] P.D. Jarvis, M. Yang and B.G. Wybourne, Generalized quasispin for supergroups, J. Math. Phys. 28 (1987) 1192.